

Algebraic entropy for differential-delay equations

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Abstract

We extend the definition of algebraic entropy to a class of differential-delay equations. The vanishing of the entropy, as a structural property of an equation, signals its integrability. We suggest a simple way to produce differential-delay equations with vanishing entropy from known integrable differential-difference equations.

1 Introduction

Algebraic entropy was introduced as a measure of complexity and a test of integrability of maps [1], an elaboration on the results of [2, 3]. It associates to any (bi)rational map over projective space an invariant quantity characterising its complexity. The drop of complexity signalled by the vanishing of algebraic entropy is the signature of integrability. The entropy usually happens to be calculable exactly, and enjoys remarkable properties, making it an object of interest *per se*. It has for example been conjectured to always be the logarithm of an algebraic integer [1] and such algebraicity property of entropies are currently under study, see for example [4].

Initially devised for systems with a finite number of degrees of freedom, the concept has been extended to systems with an infinite number of degrees of freedom, by going from maps (ordinary difference equations) to lattice equations (partial difference equations) [5, 6], and further to systems mixing differential and difference equations [7]. We present here an extension to a class of differential-delay equations.

The question of the integrability of this type of equation has already been addressed in the literature [8, 9] [10, 11], leading in particular to the concept of delay Painlevé equations, one source of such equations being the symmetry reductions of integrable lattice equations, or semi discrete equations.

Of course we have gone far away from the original definition of integrability given by Liouville for hamiltonian systems, but we wish to call integrable any system with vanishing algebraic entropy.

The equations we consider are recurrences of order k of the form

$$u(t+1) = R(u(t), u(t-1), \dots, u(t-k+1)) \quad (1)$$

with R a rational function of $u(t)$, $u(t-1)$, \dots , $u(t-k+2)$, and their derivatives, and homographic in $u(t-k+1)$, possibly depending explicitly on t in the non autonomous case. The homographic nature of (1) with respect to $u(t-k+1)$ allows to define a *birational* map from $u(t), u(t-1), \dots, u(t-k+1)$ and their derivatives to $u(t+1), u(t), \dots, u(t-k+2)$ and their derivatives.

The setting is thus very similar to the one of [7]. The crucial difference is that the derivative and the shift act on the same variable t . We will nevertheless consider the values of the unknown function u at the successive points $\dots, t-1, t, t+1, \dots$ as *independent* members of a sequence as was done in [9, 12], and write

$$u_k(t) = u(t+k).$$

Caveat: Accumulating infinitely many derivative of some u_n amounts, if the Taylor expansion of u_n converges, to giving more than one element of the sequence of u 's. We will come back to this point later.

The definition of the entropy then reproduces the one given in [7]. We use $k+1$ homogeneous coordinates to manipulate only polynomial differential expressions. Defining

$$\left[\begin{array}{lcl} u_n & = & x_1/x_{k+1} \\ u_{n-1} & = & x_2/x_{k+1} \\ & \dots & \\ u_{n-k+1} & = & x_k/x_{k+1} \end{array} \right] \quad \text{and} \quad \left[\begin{array}{lcl} u_{n+1} & = & y_1/y_{k+1} \\ u_n & = & y_2/y_{k+1} \\ & \dots & \\ u_{n-k+2} & = & y_k/y_{k+1} \end{array} \right].$$

Equation (1) can be rewritten as a differential polynomial map

$$\varphi : [x_1, \dots, x_{k+1}] \longrightarrow [y_1, \dots, y_{k+1}] \quad (2)$$

The inverse ψ is also a differential polynomial map. One may define the quantity κ_φ and κ_ψ , which are useful for the singularity analysis. Recall that κ_φ (resp. κ_ψ) are the multipliers appearing when composing φ and ψ into $\psi \cdot \varphi$ (resp $\varphi \cdot \psi$), both these products being the identity operator (see [1]).

The successive iterates produce a sequence of polynomials in the u_k 's and their derivatives. Attributing a weight 1 to all u_k and their derivatives yields the sequence $\{w_n\}$ of the weights of the iterates (Leibnitz rule ensures that all components have the same weight at all stages). Moreover the straightforward property $w_{l+m} \leq w_l + w_m$ ensures the existence of the limit

$$\epsilon = \lim_{m \rightarrow \infty} \frac{1}{m} \log(w_m). \quad (3)$$

the *algebraic entropy*.

What is remarkable is that the asymptotic behaviour of the sequence of weights is entirely determined by local features: finite subsequences contain enough information to extract the entropy. This shows itself in most cases by the existence of a finite recurrence relation on the weights (see [13] for a discussion). This is best seen when one can fit the generating function of the sequence by a rational fraction¹.

This property has two merits:

¹The fact that ϵ is the log of an algebraic integer appears there

- it allows to calculate the entropy efficiently, by exempting us from the calculation of the entire sequence $\{w_n\}$

- it also allows to consider $u(t)$ and $u(t+k)$ as independent objects, and thus to use the same method of calculation as in [7]. Since we only use a finite number of iterates, we will use derivatives only up to a finite order. We then avoid the risk of a mismatch between the value of u at some point and the value given by a Taylor expansion from some other point.

2 An explicit example

We will apply this scheme to the following equation [8]:

$$a u(t) - b \dot{u}(t) = u(t) [u(t+1) - u(t-1)] \quad (4)$$

where \dot{u} means time derivative.

The choice is motivated by multiple reasons:

- Equation (4) was obtained in [8] by a non trivial reduction of a semi-discrete equation [14] and should inherit the integrability properties of the latter. The existence of an induced Lax pair is one.
- It has rational solutions for specific values of the parameters.
- One continuous limit is Painlevé I equation.
- The Nevanlinna theory approach [15–17] was successfully applied to this equation [18].

Equation (4) defines a recurrence of order 2 translating into the map φ and its inverse ψ .

$$\varphi : [x_1, x_2, x_3] \longrightarrow [a x_1 x_3 - b (x'_1 x_3 - x_1 x'_3) + x_1 x_2, x_1^2, x_1 x_3] \quad (5)$$

$$\psi : [x_1, x_2, x_3] \longrightarrow [x_2^2, -a x_2 x_3 + b (x'_2 x_3 - x_2 x'_3) + x_1 x_2, x_2 x_3] \quad (6)$$

where prime (') means derivative.

These birational maps act on the algebra generated by $x_1, x'_1, x''_1, \dots, x_3, x'_3, x''_3, \dots$

The multipliers κ_φ and κ_ψ are just

$$\kappa_\varphi([x_1, x_2, x_3]) = x_1^3, \quad \kappa_\psi([x_1, x_2, x_3]) = x_2^3.$$

The sequence of weights of the iterates is

$$\{w_n\} = 1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, \dots \quad (7)$$

The generating function of this sequence is fitted by the rational fraction

$$g(s) = \sum_n d_n s^n = \frac{1 + 2s^3}{(1+s)(1-s)^3}, \quad (8)$$

showing quadratic growth of the weights, a standard signature of integrability.

Notice that in order to produce the sequence (7), we use arbitrary initial conditions, i.e. x_1, x_2 and x_3 which are *independent* functions of t . Recall also that what matters in (8) is the location of its poles and the order of the pole at $s = 1$.

We may perform a finer analysis of the factorisation process leading to this sequence by keeping track of the factors which are removed from the homogeneous coordinates.

Starting from

$$U_0 = [A_0, B_0, C_0],$$

we get the sequence

$$\begin{aligned} U_1 &= [A_1, A_0^2, A_0 C_0] \\ U_2 &= [A_2, A_1^2, A_0 A_1 C_0] \\ U_3 &= [A_0^2 A_3, A_2^2, A_0 A_1 A_2 C_0] \\ U_4 &= [A_1^2 A_4, A_0 A_3^2, A_1 A_2 A_3 C_0] \\ &\dots \\ U_k &= [A_{k-3}^2 A_k, A_{k-4} A_{k-1}^2, A_{k-3} A_{k-2} A_{k-1} C_0] \end{aligned}$$

where the A_k 's are polynomials in the initial conditions U_0 and their derivatives.

We see that the factorisation pattern stabilises after the fourth iterate, which is actually the first order at which a non trivial factorisation happens, and we will suppose it remains stable.

Denote by X_k the first component of U_k . We get the image U_{k+1} of U_k , in two steps. First evaluate $\varphi(U_k)$ using the homogeneous expression (5). Any common factor to the three components is then removed. Call f_{k+1} this factor. The previous sequence indicates that beyond the fourth iterate:

$$f_n \cdot f_{n-3}^2 = X_{n-4}^3.$$

This implies the finite recurrence relation

$$w_{n+4} - 2 w_{n+3} + 2 w_{n+1} - w_n = 0.$$

The characteristic polynomial of this recurrence relation is

$$s^4 - 2 s^3 + 2 s - 1 = (s + 1) (s - 1)^3,$$

where we recover the denominator of the generating function (8), and the quadratic growth of the weights:

$$w_n = \frac{1}{8} \left(6 n^2 + 9 - (-1)^n \right).$$

Remark: The fact that factors of increasing degree appear when we proceed with the iteration means that the components of U_k verify *relations linking consecutive points*, as

$$A_k A_{k+3} - b A'_{k+1} A_{k+2} C_0 = P A_{k+1},$$

for some P . This factorisation property yields

$$f_k = A_{k-4}^3.$$

3 Departing from integrability

It is interesting to see how the picture changes if we deviate from (4), taking for example

$$a u(t) - b \dot{u}(t) = u(t) [u(t+1) - \lambda u(t-1)] \quad (9)$$

For generic λ we get the sequence

$$\{w_n\} = 1, 2, 4, 8, 16, 32, 64, \dots \quad (10)$$

that is to say no factorisation, and algebraic entropy $\epsilon = \log(2)$.

The proof is straightforward: the multiplier κ_φ is slightly modified.

$$\kappa_\varphi([x_1, x_2, x_3]) = \lambda x_1^3.$$

The successive images of the surface $\kappa_\varphi = 0$ are the points $[-1, 0, 0]$, $[-1, 1, 0]$, $[1 - \lambda, 1, 0]$, $[1 + \lambda, 1 - \lambda, 0]$, $[(1 - \lambda)\lambda, 1 + \lambda, 0]$, $[1 + \lambda, 1 - \lambda, 0]$ and so on. If $\lambda \neq \pm 1$ we never meet singular points, so there is no drop of the weight.

Notice that the special value $\lambda = -1$ brings in some drop of the weights, with the sequence

$$\{w_n\} = 1, 2, 4, 8, 16, 30, 56, 104, 192, \dots \quad (11)$$

fitted by the rational generating function

$$g(s) = \sum_k d_k s^k = \frac{1 + s^4}{(1 - s)(1 - s - s^2 - s^3)} \quad (12)$$

non vanishing entropy, no integrability.

4 Folding

We have shown, on a specific example, that a differential-delay equation, which was considered to have the other features of integrability, also “pass the algebraic entropy test”.

The key point allowing the calculation of the entropy was to consider the values of the unknown function u at different points $\dots, t-1, t, t+1, \dots$ as members of a sequence of function verifying a simple recurrence relation.

Many differential-delay equations will then pass the algebraic entropy test. Starting from any differential-difference system of equations with vanishing entropy we may turn it to a differential-delay system with zero entropy, that is to say a candidate to integrability.

Take for example the Ablowitz-Ladik system [19]:

$$\dot{q}_n = q_{n+1} - 2 q_n + q_{n-1} + q_n r_n (q_{n+1} + q_{n-1}) \quad (13)$$

$$-\dot{r}_n = r_{n+1} - 2 r_n + r_{n-1} + q_n r_n (r_{n+1} + r_{n-1}). \quad (14)$$

This is an integrable semi-discretisation of the nonlinear Schrödinger equation. The time variable t remained continuous, and the space variable x is discretised to n . Folding the discrete dimension n over the continuous one t leads to the differential-delay system

$$\dot{q}(t) = q(t+1) - 2 q(t) + q(t-1) + 2 q(t) r(t) (q(t-1) + q(t+1)) \quad (15)$$

$$-\dot{r}(t) = r(t+1) - 2 r(t) + r(t-1) + 2 r(t) q(t) (r(t-1) + r(t+1)) \quad (16)$$

which will pass the entropy test, since the system (13,14) does [7], *because the calculations of both entropies are identical*.

Remark: The case $a = 0$ of equation (4) [eq 18 of Ref [8]] can be obtained by this folding operation from the original equation (4) of Ref. [14]. This also means that the latter has vanishing entropy as a semi-discrete equation.

Many more can be written using this rather rustic reduction, and of course more sophisticated ones, in particular from the various symmetries of integrable lattice equations for which the entropy has already been shown to vanish in [7]. We will get in this manner a number of interesting autonomous and non-autonomous differential-delay equations.

5 Conclusion

The vanishing of the algebraic entropy is a structural property of an equation. The merit of the entropy test is that we may perform it without knowing any of the solutions. Solving the equations implies to go beyond the discrete aspects, and to solve an interpolation problem, which is hard, and usually requires some input on the nature of the solution (domains of continuity, differentiability, analyticity, ...). The entropy test allows to choose a limited set of equations of interest.

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